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A_α-SPECTRA OF DUPLICATION CORONAS

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Abstract: The convex combination of diagonal matrix $D(G)$ and adjacency matrix $A(G)$ defined as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$ for $\alpha \in [0,1]$ is indeed a good choice to merge A-spectra and Q-spectra theories of graphs. In this work, we discuss A_α-eigenvalues of duplication corona, duplication edge corona, duplication neighbourhood corona and vertex complemented corona. Motivated by these operations, we have defined duplication vertex complemented corona operation and obtained corresponding A_α-eigenvalues. We have also suggested some infinite family of graphs based on these findings.

Keywords and Phrases: A_α-spectra, duplication edge corona, duplication neighbourhood corona, vertex complemented corona, duplication vertex complemented corona.

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1. Introduction

In this work we have only considered simple and connected graphs. Let $G = (V, E)$ be a graph where $V = V(G)$ and $E = E(G)$ represents vertex set and edge set of G respectively. The total number of vertices and edges denoted by $|V(G)|$ and $|E(G)|$ are respectively called as order and size of G. An edge between two vertices v_i and v_j is represented by (v_i, v_j) and d_v i.e., degree of v gives total number of vertices adjacent to v. For a graph G, the complement of G have all the edges not in G but in its corresponding complete graph with same order. Also neighbourhood of a vertex v denoted by $N(v)$ defines the set of adjacent vertices of v.

The adjacency matrix $A(G)$ is one of the most primitive matrix associated with a graph where $A(G) = [a_{ij}]$ such that $a_{ij} = 1$ if v_i is adjacent to v_j and 0 otherwise. The set of all eigenvalues of $A(G)$ together with its multiplicities is said to be the spectrum of G. Over the years various modified adjacency like matrices such as Laplacian matrix $L(G) = D(G) - A(G)$ (where $D(G)$ is corresponding diagonal matrix), signless Laplacian matrix $Q(G) = D(G)+A(G)$, distance signless Laplacian matrix $Q^{D(G)}$ etc. are proposed and their corresponding spectra together with various other properties are extensively studied. In literature, the spectra of $L(G)$ and $Q(G)$ are known as L-spectrum and Q-spectrum of G. In 2017, Nikiforov [11] proposed a new matrix which is the convex combination of $D(G)$ and $A(G)$ where

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G), \qquad 0 \leq \alpha \leq 1$$

and put forwarded different aspects of study to extend the properties of constituent matrices to A_α-matrices. The set of all eigenvalues of $A_\alpha(G)$ is referred to as A_α-spectrum of G. In the last few years, extensive volume of research has been carried out to attain and explore the properties of A_α in different dimensions and involving different parameters. Two graphs with same set of eigenvalues are termed as cospectral graphs. Similarly two graphs are A_α-cospectral if they have same A_α-spectrum.

In literature, defining various operations on graphs and finding its different spectra (say L-spectra, Q-spectra etc.) are considered as interesting topics to discuss and discover. In recent years, various variants of corona product of graphs say edge corona, neighbourhood corona, R-vertex corona, R-vertex edge corona etc. are introduced and spectra are studied (some references are [3, 9, 10]). Motivated by these, in this paper we have found the A_α-spectra of various types of duplication corona operations and suggested some methods to generate infinitely many family of A_α-cospectral graphs. For some recent works on A_α-matrices, one can visit papers [12, 13, 14].

Now we discuss some known results and lemmas that helped us to attain the desired results. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of order $m \times n$ and $p \times q$ respectively. Then the Kronecker product [4] of A and B denoted by $A \otimes B$ is the $mp \times nq$ order matrix obtained by substituting each a_{ij} of A by $a_{ij}B$. Also, the property of Kronecker product shows that $(X \otimes Y)(Z \otimes W) = XZ \otimes YW$. Throughout the content, we have used the symbol J_n to represent an $n \times n$ matrix with all entries 1 and notation e is used to denote a column vector of compatible size with all entries 1. $Spec(G)$ refers the set of all eigenvalues of G with respective algebraic multiplicities. For a matrix A, $f(A, x) = det(xI - A(G))$ represents the characteristic polynomial of A.

Lemma 1.1. [16] *Let M_1, M_2, M_3 and M_4 be matrices such that M_4 is non-singular. Then*

$$\begin{vmatrix} M_1 & M_2 \\ M_3 & M_4 \end{vmatrix} = |M_4||M_1 - M_2{M_4}^{-1}M_3|$$

where $M_1 - M_2{M_4}^{-1}M_3$ is called the Schur complement [16] *of M_4.*

Lemma 1.2. [3, 10] *For a square matrix M of order n, M- coronal denoted by $\Gamma_M(x)$ is defined to be the sum of the entries of the matrix $(xI_n - M)^{-1}$ i.e., $\Gamma_M(x) = e^T(xI_n - M)^{-1}e$, where e is the column vector of size n whose all entries equal to 1.*

Lemma 1.3. [3] *Let M be a graph matrix of order n. If each row sum of M is equal to some constant r, then $\Gamma_M(x) = \dfrac{n}{x - r}$.*

Lemma 1.4. [9] *Let α be a real number, A an $n \times n$ real matrix, I_n the identity matrix of size n and J_n the $n \times n$ matrix with all entries equal to 1. Then*

$$det(xI_n - A - \alpha J_n) = (1 - \alpha\Gamma_{A(G)}(x)) \times det(xI_n - A).$$

Lemma 1.5. [5] *Let*

$$A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$$

be a 2×2 block matrix. Then the eigenvalues of A are those of $A_0 + A_1$ together with those of $A_0 - A_1$.

Theorem 1.6. [8] *Let G and G' be two graphs such that $Spec(A_\alpha(G)) = Spec(A_\alpha(G'))$ for $\alpha \in [0, 1]$. Then*

a $|V(G)| = |V(G')|$;

b $|E(G)| = |E(G')|$;

c *If* G *is* r*-regular, then* G' *is* r*-regular.*

The following definitions are the basis of our findings:

Definition 1.7. (Duplication Graph) [7] *For a graph* G *with vertex set* $V(G) = \{v_1, \cdots, v_n\}$ *and edge set* $E(G)$*, the Duplication graph* $Du(G)$ *of* G *is a bipartite graph with vertex partition sets* $U = \{u_1, \cdots, u_n\}$ *and* $V = \{v_1, \cdots, v_n\}$*, where* u_iv_j *is an edge in* $Du(G)$ *if and only if* v_iv_j *is an edge in* G *and then deleting the edges only in* G*.*

Definition 1.8. (Duplication Corona) [1] *The Duplication Corona* $G \boxminus F$ *of two graphs* G *and* F *is the graph obtained by taking one copy of* $Du(G)$ *and* $|V(G)|$ *copies of* F *and then joining the vertex* v_i *of* $Du(G)$ *to every vertex in the* i^{th} *copy of* F*.*

Definition 1.9. (Duplication Neighbourhood Corona) [1] *The duplication neighbourhood corona* $G \boxtimes F$ *of two graphs* G *and* F *is the graph obtained by taking one copy of* $Du(G)$ *and* $|V(G)|$ *copies of* F *and then joining the neighbours of the vertex* v_i *of* $Du(G)$ *to every vertex in the* i^{th} *copy of* F*.*

Definition 1.10. (Duplication Edge Corona) [1] *The duplication edge corona operation* $G \boxplus F$ *of two graphs* G *and* F *is the graph obtained by taking one copy of* $Du(G)$ *and* $|E(G)|$ *copies of* F*, and then joining a pair of vertices* v_i *and* v_j *of* $Du(G)$ *to every vertex in the* k^{th} *copy of* F *whenever* $(v_i, v_j) = e_k \in E(G)$*.*

Definition 1.11. (Vertex Complemented Corona) [15] *Let* G *be a graph with vertex set* $V(G) = \{v_1, \cdots, v_n\}$ *and let* $\vec{H} = (H_1, \cdots, H_n)$ *be an* n*- tuple of graphs. The vertex complemented corona* $G\tilde{\circ}\vec{H}$ *is formed by taking the disjoint union of* G *and* $H_1, \cdots, H_n$ *with each* H_i *corresponding to the vertex* v_i *and then joining every vertex in* H_i *to every vertex in* $V(G) \setminus \{v_i\}$ *for* $i \in \{1, \cdots, n\}$*.*

Throughout the paper, we use the word multiplicity of an eigenvalue to mean the algebraic multiplicity of the eigenvalue.

2. Main Findings

In this section we have discussed A_α- eigenvalues of duplication corona, duplication neighbourhood corona, duplication edge corona, vertex complemented corona and duplication vertex complemented corona subjected to some conditions. We report some novel results related to vertex complemented, and duplication vertex complemented corona of graphs, while some of the results presented in this section were independently derived by Brondani et al., as reported in [2].

2.1. A_α-Spectra of Duplication Corona

It is to note that for any arbitrary graphs G and F, the order and size of duplication corona $G \boxminus H$ are $2|V(G)|+|V(G)||V(F)|$ and $2|E(G)|+|E(F)||V(G)|+|V(G)||V(F)|$, respectively.

Theorem 2.1. *Let G be a r-regular graph with n vertices and let F be any graph with m vertices, then the characteristic polynomial of $A_\alpha(G \boxminus F)$ is given by*

$$\prod_{i=1}^{m}(x-\alpha-\lambda_i(A_\alpha(F))^n \prod_{i=1}^{n}[x-\alpha r-\alpha m-(\alpha-1)^2\Gamma_{A_\alpha(F)}(x-\alpha)-(\alpha-1)^2\lambda_i(A(G))].$$

Proof. Labelling the vertices of G and F in a suitable manner, the A_α-matrix of $G \boxminus F$ can be obtained as follows:

$$A_\alpha(G \boxminus F) = \begin{bmatrix} I_n \otimes (A_\alpha(F) + \alpha I_m) & 0 & (1-\alpha)I_n \otimes e \\ 0 & \alpha D(G) & (1-\alpha)A(G) \\ (1-\alpha)I_n \otimes e^T & (1-\alpha)A(G) & \alpha D(G) + m\alpha I_n \end{bmatrix}.$$

Therefore the characteristic polynomial of $A_\alpha(G \boxminus F)$ is given by

$$f((A_\alpha(G \boxminus F), x) = det \begin{bmatrix} I_n \otimes ((x-\alpha)I_m - A_\alpha(F)) & 0 & (\alpha-1)I_n \otimes e \\ 0 & xI_n - \alpha D(G) & (\alpha-1)A(G) \\ (\alpha-1)I_n \otimes e^T & (\alpha-1)A(G) & xI_n - \alpha D(G) - m\alpha I_n \end{bmatrix}.$$

Now, if G is a r-regular graph, then by using Lemma 1.1 we have

$$\begin{aligned} &= det \begin{bmatrix} I_n \otimes ((x-\alpha)I_m - A_\alpha(F)) & 0 & (\alpha-1)I_n \otimes e \\ 0 & (x-\alpha r)I_n & (\alpha-1)A(G) \\ (\alpha-1)I_n \otimes e^T & (\alpha-1)A(G) & (x-\alpha r-\alpha m)I_n \end{bmatrix} \\ &= det(I_n \otimes ((x-\alpha)I_m - A_\alpha(F))) \times \\ &\quad det\left(S - \begin{bmatrix} 0 \\ (\alpha-1)I_n \bigotimes e^T \end{bmatrix} [I_n \otimes ((x-\alpha)I_m - A_\alpha(F))]^{-1} \begin{bmatrix} 0 & (\alpha-1)I_n \otimes e, \end{bmatrix}\right) \end{aligned}$$

where

$$S = \begin{bmatrix} (x-\alpha r)I_n & (\alpha-1)A(G) \\ (\alpha-1)A(G) & (x-\alpha r-\alpha m)I_n \end{bmatrix}.$$

Hence the characteristic polynomial of $A_\alpha(G \boxminus F)$ is

$$\begin{aligned}
&= det(I_n \otimes ((x-\alpha)I_m - A_\alpha(F))) \\
&\quad \times det \begin{bmatrix} (x-\alpha r)I_n & (\alpha-1)A(G) \\ (\alpha-1)A(G) & (x-\alpha r-\alpha m-(\alpha-1)^2\Gamma_{A_\alpha(F)}(x-\alpha))I_n \end{bmatrix} \\
&= \prod_{i=1}^{m} (x-\alpha-\lambda_i(A_\alpha(F)))^n \times det\,((x-\alpha r)I_n) \times \\
&\quad det\left((x-\alpha r-\alpha m-(\alpha-1)^2\Gamma_{A_\alpha(F)}(x-\alpha))I_n-(\alpha-1)^2A(G)^2\frac{1}{(x-\alpha r)}\right) \\
&= \prod_{i=1}^{m} (x-\alpha-\lambda_i(A_\alpha(F)))^n \\
&\quad \times \prod_{i=1}^{n}\left(\left(x-\alpha r-\alpha m-(\alpha-1)^2\Gamma_{A_\alpha(F)}(x-\alpha)\right)(x-\alpha r)-(\alpha-1)^2\lambda_i^2(G)\right).
\end{aligned}$$

Corollary 2.2. *If G and F be r- and s-regular graphs with n and m vertices respectively, then the eigenvalues of $G \boxminus F$ are given as follows:*

i . $\alpha + \lambda_i(A_\alpha(F))$ with multiplicity n for $i \in \{2, \cdots, m\}$.

ii . For each $i \in \{2, \cdots, n\}$, the roots of cubic polynomial

$$x^3 - x^2(\alpha+s+\alpha r+\alpha m) + x[2\alpha r(\alpha+s) + \alpha^2r^2 + \alpha^2mr + \alpha m(\alpha+s) - m(\alpha-1)^2 - (\alpha-1)^2\lambda_i^2(G)] - \alpha^2r^2(\alpha+s) - \alpha^2mr(\alpha+s) + m\alpha r(\alpha-1)^2 + (\alpha-1)^2(\alpha+s)\lambda_i^2(G).$$

Proof. If F is a s-regular graph, then s is an eigenvalue of $A_\alpha(F)$. Hence by using Lemma 1.3 in Theorem 2.1, the results automatically follow.

Corollary 2.3. *If G_1 is r-regular and G_2 be any graph such that G_1 and G_2 are A_α- cospectral i.e., $Spec(A_\alpha(G_1)) = Spec(A_\alpha(G_2))$. Then for any arbitrary graph F, $G_1 \boxminus F$ and $G_2 \boxminus F$ are A_α-cospectral.*

Proof. The result follows from Theorems 1.6 and 2.1.

Corollary 2.4. *For any arbitrary graph G, if F_1 and F_2 are A_α-cospectral such that $\Gamma_{A_\alpha(F_1)} = \Gamma_{A_\alpha(F_2)}$, then $G \boxminus F_1$ and $G \boxminus F_2$ are A_α-cospectral.*

2.2. A_α-Spectra of Duplication Neighbourhood Corona

From the definition it is clear that number of vertices and edges of duplication neighbourhood corona $G \boxtimes F$ are $2|V(G)| + |V(G)||V(F)|$ and $2|E(G)| + |V(G)||E(F)| + |V(F)|\sum_{i=1}^{n}|N(v_i)|$ respectively.

Theorem 2.5. *Let G be a r- regular graph with n vertices and let F be any graph with m vertices. Then the characteristic polynomial of $G \boxtimes F$ is given by*

$$\prod_{i=1}^{m}(x - r\alpha - \lambda_i(A_\alpha(F))^n \prod_{i=1}^{n}(((x - \alpha r(m+1)) - (\alpha-1)^2\Gamma_{A_\alpha(F)}$$
$$(x-\alpha r)\lambda_i^2(G))(x-\alpha r) - (\alpha-1)^2\lambda_i^2(G)).$$

Proof. Labelling the vertices of G and F in a suitable manner, $A_\alpha(G \boxtimes F)$ can be obtained as follows:

$$A_\alpha(G \boxplus F) = \begin{bmatrix} I_n \otimes (A_\alpha(F) + \alpha r I_m) & 0 & (1-\alpha)A(G) \otimes e \\ 0 & \alpha r I_n & (1-\alpha)A(G) \\ (1-\alpha)A(G) \otimes e^T & (1-\alpha)A(G) & \alpha r(m+1)I_n \end{bmatrix}$$

Proceeding similarly as above, we can obtain the desired result.

The following corollary follows directly from Theorem 2.5.

Corollary 2.6. *If G and F be r and s regular graphs with n and m vertices respectively, then the spectrum of $G \boxtimes F$ consists of the following:*

i . $\alpha r + \lambda_i(A_\alpha(F))$ with multiplicity n for $i \in \{2, \cdots, m\}$.

ii . For each $i \in \{2, \cdots, n\}$, the roots of cubic polynomial

$$x^3 - x^2(3\alpha r + \alpha r m + s) + x(3\alpha^2 r^2 + 2\alpha^2 r^2 m + 2\alpha r s + \alpha r m s - (m+1)$$
$$(\alpha-1)^2\lambda_i^2(G)) + (\alpha-1)^2\lambda_i^2(G)(\alpha r m + \alpha r + s) - \alpha^2 r^2(m+1)(\alpha r + s).$$

Corollary 2.7. *If G_1 is r- regular and G_2 be any graph such that G_1 and G_2 are A_α-cospectral. Then for any arbitrary graph F, $G_1 \boxtimes F$ and $G_2 \boxtimes F$ are A_α-cospectral.*

Corollary 2.8. *For any arbitrary graph G, if F_1 and F_2 are A_α-cospectral such that $\Gamma_{A_\alpha(F_1)} = \Gamma_{A_\alpha(F_2)}$, then $G \boxtimes F_1$ and $G \boxtimes F_2$ are A_α-cospectral.*

2.3. A_α-Spectra of Duplication Edge Corona

Definition 1.10 suggests that the order and size of duplication edge corona $G \boxplus F$ are $2|V(G)| + |E(G)||V(F)|$ and $2|E(G)| + |E(F)||E(G)| + 2|E(G)||V(F)|$ respectively. Here, the symbol B represents incidence matrix of graph G.

Theorem 2.9. *If G be a r-regular graph with n_1 vertices and m_1 edges and if F be any graph with n_2 vertices, then the characteristic polynomial of A_α-matrix of*

$G \boxplus F$ is given by

$$\prod_{i=1}^{n_1}\left[\{x-\alpha r(n_2+1)-(\alpha-1)^2(\lambda_i(G)+r)\Gamma_{A_\alpha(F)}(x-\alpha r)\}(x-\alpha r)-(\alpha-1)^2\lambda_i^2(G)\right]$$
$$\times\prod_{i=1}^{n_2}(x-\alpha r-\lambda_i(A_\alpha(F)))^{m_1}$$

Proof. By suitable labelling of vertices and edges of the graphs G and F, we get

$$A_\alpha(G\boxplus F)=\begin{bmatrix} I_{m_1}\otimes(\alpha r I_{n_2}+A_\alpha(F)) & 0 & (1-\alpha)B^T\otimes e_{n_2} \\ 0 & \alpha r I_{n_1} & (1-\alpha)A(G) \\ (1-\alpha)B\otimes e_{n_2}^T & (1-\alpha)A(G) & \alpha r(n_2+1)I_{n_1}\end{bmatrix}$$

Therefore, after routine calculations like in Theorems 2.1 and 2.5, we obtain the required result.

The following corollary on having infinite family of A_α-cospectral graphs can be automatically derived from the above theorem.

Corollary 2.10. *If G be a r-regular graph with n_1 vertices and m_1 edges and let F be a s-regular graph with n_2 vertices, then the eigenvalues of $A_\alpha(G\boxplus F)$ consists of*

i . $\alpha r+\lambda_i(A_\alpha(F))$ with multiplicity m_1 for $i\in\{2,\cdots,n_2\}$.

ii . $\alpha r+s$ with multiplicity m_1-n_1.

iii . For each $i\in\{2,\cdots,n_1\}$, the roots of cubic polynomial

$$x^3-x^2\{s+\alpha r(n_2+1)\}+x\{\alpha^2r^2(2n_2+2)+\alpha^2r^2+\alpha rs-(\alpha-1)^2(\lambda_i(G)+r)n_2+$$
$$(\alpha-1)^2\lambda_i^2(G)\}-\alpha^2r^2(\alpha r+s)(n_2+1)+\alpha r(\alpha-1)^2(\lambda_i(G)+r)n_2+(\alpha r+s)(\alpha-1)^2\lambda_i^2(G).$$

Corollary 2.11. *If G_1 is r-regular and G_2 be any graph such that G_1 and G_2 are A_α-cospectral. Then for any arbitrary graph F, $G_1\boxplus F$ and $G_2\boxplus F$ are A_α-cospectral.*

Corollary 2.12. *For any arbitrary graph G, if F_1 and F_2 are A_α-cospectral such that $\Gamma_{A_\alpha(F_1)}=\Gamma_{A_\alpha(F_2)}$, then $G\boxplus F_1$ and $G\boxplus F_2$ are A_α-cospectral.*

2.4. A_α-Spectra of Vertex Complemented Corona

It is to note that the order of vertex complemented corona $G\tilde{\circ}\vec{H}$ is $|V(G)|+\sum_{i=1}^n|V(H_i)|$ and size is $|E(G)|+\sum_{i=1}^n|E(H_i)|+(|V(G)|-1)\sum_{i=1}^n|V(H_i)|$. The vertices of $G\tilde{\circ}\vec{H}$ can be labelled as below:

$$V(G\tilde{\circ}\vec{H})=\{(v,0):v\in V(G)\}\cup_{i=1}^n\{(v_i,w):v_i\in V(G),w\in V(H_i)\}$$

and adjacency defined in Definition 1.11 can also be viewed as

$$(v_i, w) \sim (v_j, w') \leftrightarrow \begin{cases} w = w' = 0 \quad \text{and} \quad v_i \sim v_j \ \text{ in } \ G \\ \text{or} \\ v_i = v_j \quad \text{and} \quad w \sim w' \ \ in \ \ H_i \\ \text{or} \\ v_i \neq v_j \quad \text{and just one of w and } w' \text{ is } 0 \end{cases}$$

Theorem 2.13. *Let G be an r-regular connected graph with order $n \geq 2$ and let $\vec{H} = (H_1, \cdots, H_n)$ be an n-tuple of each k-regular graphs with $|V(H_i)| = m \geq 1$ for $i \in \{1, \cdots, n\}$. Suppose G has eigenvalues $r = \lambda_0 > \cdots \lambda_p$ with multiplicities $r_0 = 1, r_1, \cdots, r_p$. Then the A_α- spectrum of $G\tilde{\circ}\vec{H}$ consists of the following:*

- *i . $\alpha(n-1) + k$ with multiplicity $(\sum_{i=1}^{n} s_i) - n$, where s_i denotes multiplicity of eigenvalue $\alpha(n-1) + k$ of H_i.*
- *ii . $\alpha(n-1) + \mu$ with multiplicities r_μ, where $\alpha(n-1) + \mu$ is an eigenvalue of H_i with multiplicity r_μ, consisting of all eigenvalues of H_i except $\alpha(n-1)+k$ for each $i \in \{1, \cdots, n\}$.*
- *iii . Roots of the quadratic polynomial $x^2 - x\{\alpha(n + mn - m - 1) + r + k\} + \alpha(n-1)\{m\alpha(n-1) + km + r\} - m(\alpha-1)^2(n-1)^2 + rk$ with multiplicity 1.*
- *iv . Roots of the quadratic polynomial $x^2 - x\{\alpha(n + mn - m - 1) + \lambda_j + k\} + \alpha(n-1)\{m\alpha(n-1) + km + \lambda_j\} - m(\alpha-1)^2 + \lambda_j k$ with multiplicity r_j for $j \in \{1, \cdots, p\}$.*

Proof. Arranging the vertices in a suitable way, the A_α- matrix of $G\tilde{\circ}\vec{H}$ can be written as follows:

$$G\tilde{\circ}\vec{H} = \begin{bmatrix} A_\alpha(G) + \alpha m(n-1)I_n & (1-\alpha)(J_n - I_n) \otimes e^T \\ \\ (1-\alpha)(J_n - I_n)^T \otimes e & I_n \otimes A_\alpha(H_i) + \alpha(n-1)I_m \end{bmatrix}$$

Therefore the characteristic polynomial of $A_\alpha(G\tilde{\circ}\vec{H})$ is

$$\begin{aligned} f(A_\alpha(G\tilde{\circ}\vec{H}), x) = det\left[I_n \otimes ((x - \alpha(n-1))\, I_m - A_\alpha(H_i))\right] \times \\ det\left[(x - m\alpha(n-1)I_n - A_\alpha(G) - (\alpha-1)^2 M \otimes e^T S^{-1} M^T \otimes e\right] \end{aligned}$$

where $M = J_n - I_n$ and $S = I_n \otimes ((x - \alpha(n-1))\, I_m - A_\alpha(H_i)$. Hence by using Lemmas 1.3 and 1.4 and simplifying, we get the required result.

2.5. A_α-Spectra of Duplication Vertex Complemented Corona

Motivated by the earlier graph operations, we have proposed the following graph operation and computed its A_α-eigenvalues for some cases.

Definition 2.14. (Duplication vertex complemented corona) *The Duplication vertex complemented corona $G\tilde{\boxdot}F$ of two graphs G and F is the graph obtained by taking one copy of $Du(G)$ and $|V(G)|$ copies of F and then joining every vertex in i^{th} copy of F to every vertex in $V(G)\setminus\{v_i\}$.*

Here the number of vertices and edges of $G\tilde{\boxdot}F$ are $2|V(G)|+|V(G)||V(F)|$ and $2|E(G)|+|V(G)||E(F)|+|V(F)|(|V(G)|-1)|V(G)|$.

Theorem 2.15. *Let G be a r-regular and F be a s-regular graph, then the eigenvalues of $G\tilde{\boxdot}F$ are*

i . $\lambda_i(F)$ for $i \in \{2,\cdots,m\}$ with multiplicity n.

ii . Roots of the cubic polynomial $x^3-x^2s-x(m+\lambda_i^2(G))+s\lambda_i^2(G)$ for each $i\in\{2,\cdots,n\}$

iii Roots of the cubic polynomial $x^3-x^2s-x\{m+r^2+mn(n-2))\}+sr^2$.

Proof. From observation, the adjacency matrix associated with $G\tilde{\boxdot}F$ can be obtained as follows:

$$A(G\tilde{\boxdot}F)=\begin{bmatrix} I_n\otimes A(F) & 0 & (J_n-I_n)^T\otimes e\\ 0 & 0 & A(G)\\ (J_n-I_n)\otimes e^T & A(G) & 0\end{bmatrix}$$

Therefore, after simplification by using Lemmas 1.1 and 1.3, we get

$$f(G\tilde{\boxdot}F,x)=det(I_n\otimes(xI_m-A(F)))\times A$$

where

$$A=det\begin{bmatrix} xI_n & -A(G)\\ -A(G) & xI_n-((n-2)J_n+I_n)\dfrac{m}{x-s}\end{bmatrix}$$

Now,

$$A=det(xI_n)\times det\left[\{(x-\frac{m}{x-s})I_n-\frac{m(n-2)}{x-s}\}-A^2(G)\frac{1}{x}\right]$$
$$=\left\{1-\frac{mx(n-2)}{x-s}\Gamma_{A^2(G)}\left(x(x-\frac{m}{x-s})\right)\right\}\times det\left\{x(x-\frac{m}{x-s})I_n-A^2(G)\right\}$$
$$=\left\{1-\frac{mx(n-2)}{x-s}\times\frac{n}{x(x-\dfrac{m}{x-s})-r^2}\right\}\times\frac{1}{(x-s)^n}\times det\left[x\{x(x-s)-m\}I_n-A^2(G)(x-s)\right]$$

Therefore, the characteristic polynomial of $G\tilde{\boxdot}F$ is

$$f(G\tilde{\boxdot}F,x)=\left[\{x(x(x-s)-m)-r^2(x-s)\}-mnx(n-2)\right]\times\prod_{i=2}^{m}(x-\lambda_i(F))^n$$
$$\times\prod_{i=2}^{n}\{x(x(x-s)-m)-\lambda_i^2(G)(x-s)\}$$

and hence the theorem.

Theorem 2.16. *Let G be a r-regular and F be a s-regular graph, then the A_α-eigenvalues of $G\tilde{\boxdot}F$ are*

i . $\alpha(n-1)+\lambda_i(A_\alpha(F))$ for $i\in\{2,\cdots,m\}$ with multiplicity n.

ii . Roots of the cubic polynomial
$(x-\alpha r)\{(x-\alpha m(n-1)-\alpha r)(x-\alpha(n-1)-s)-(\alpha-1)^2m\}-(\alpha-1)^2(x-\alpha(n-1)-s)\lambda_i^2(G)$ *for $i\in\{2,\cdots,n\}$*

iii Roots of the cubic polynomial
$(x-\alpha r)\{(x-\alpha m(n-1)-\alpha r)(x-\alpha(n-1)-s)-(\alpha-1)^2m\}-(\alpha-1)^2(x-\alpha(n-1)-s)r^2-(x-\alpha r)(\alpha-1)^2mn(n-2)$.

Proof. If G be r-regular and F be s-regular graph, then A_α-matrix associated with $G\tilde{\boxdot}F$ can be written as

$$A_\alpha(G\tilde{\boxdot}F)=\begin{bmatrix} I_n\otimes(A_\alpha(F)+\alpha(n-1)I_m) & 0 & (1-\alpha)(J_n-I_n)^T\otimes e\\ 0 & \alpha rI_n & (1-\alpha)A(G)\\ (1-\alpha)(J_n-I_n)\otimes e^T & (1-\alpha)A(G) & (\alpha r+\alpha m(n-1))I_n\end{bmatrix}$$

Hence with the routine work as done above, the results follows.

From this theorem, we can suggest some infinite family of A_α- cospectral graphs.

Corollary 2.17. *Let G be a regular graph and F_1 and F_2 have same regularity such that they are A_α-cospectral. Then $G\tilde{\boxdot}F_1$ and $G\tilde{\boxdot}F_2$ are also A_α-cospectral.*

Corollary 2.18. *Let G_1 and G_2 be regular cospectral graph such that $\Gamma_{A^2(G_1)}=\Gamma_{A^2(G_2)}$. Then for any arbitrary regular graph F, $G_1\tilde{\boxdot}F$ and $G_2\tilde{\boxdot}F$ are A_α-cospectral.*

Remark 2.19. *If G be a complete graph and F be any arbitrary connected graph, then the A_α-spectrum of duplication neighbourhood corona and duplication vertex complemented corona are same.*

3. Concluding Remark

In this paper, we have discussed A_α-spectra of three various duplication corona operations on a graph with another graph. Later we have defined duplication vertex complemented corona operation and investigated its spectrum explicitly. Also we have found some infinite family of graphs based on these results.

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References

[1] Adiga, C., Rakshith, B. R. and Subba Krishna, Spectra of Some New Operations and Some New Classes of Integral Graphs, Iranian Journal of Mathematical Sciences and Informatics, 13 (2018), 51-65.

[2] Brondani, A. E., França, Francisca A. M. and Lima, Rayan G. O. J., A_α-Spectrum of duplicate corona operations in graphs, Proyecciones Journal of Mathematics, 42 (2023), 967-982.

[3] Cui, S. Y. and Tian, G. X., The Spectrum and the signless Laplacian spectrum of coronae, Linear Algebra Appl., 437 (2012), 1692-1703.

[4] Cvetković, D. M., Doob, M. and Sachs, H., Spectra of Graphs-Theory and Application, third ed., Johann Ambrosius Barth, Heidelberg, 1995.

[5] Davis, P. J., Circulant Matrices, John Wiley and Sons, 1979.

[6] Horn, R. A. and Johnson, C. R., Matrix Analysis, Cambridge University, Press, 1987.

[7] Indulal, G. and Vijaykumar, A., On a pair of equienergetic graphs, MATCH Commun. Math. Comput. Chem., 55 (2006), 83-90.

[8] Lin, H., Liu, X. and Xue, J., Graphs Determined by their $A_\alpha-$ Spectra, Discrete Mathematics, 342 (2019), 441-450.

[9] Liu, X. and Zhang, Z., Spectra of sub-division join and sub-division edge join of two graphs, B. Malays. Math. Sci. So., 42 (2019), 15-31.

[10] McLeman, C. and McNicholas, E., Spectra of coronae, Linear Algebra Appl., 435 (2011), 998-1007.

[11] Nikiforov, V., Merging the A- and Q- spectral theories, Appl. Anal. Discrete Math., 11 (2017), 81-107.

[12] Pirzada, S., Rather, Bilal A., Shaban, R. U. and Chishti, T. A., On the Sum of the Powers of A_α Eigenvalues of Graphs and A_α-energy Like Invariant, Bol. Soc. Paran. Mat., 40 (2022), 1-12.

[13] Pirzada, S., Two upper bounds on the A_α- spectral radius of a connected graph, Communications in Combinatorics and Optimization., 7(1) (2022), 53-57.

[14] Rather, Bilal A., Ganie, H. A. and Pirzada, S., On A_α spectrum of joined union of graphs and its application to power graph of finite group, Journal of Algebra and its Applications, 22(12) (2023) 2350257.

[15] Wang, J. and Liu, X., State transfer in vertex complemented coronas, Discrete Mathematics, 321 (2022), 165-178.

[16] Zhang, F. Z., The Schur Complement and its Applications, Springer, New York, 2005.

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